ISRAEL JOURNAL OF MATHEMATICS, Vol. 30, Nos. 1-2, 1978

γ-RAMSEY AND γ-INEFFABLE CARDINALS[†]

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ABSTRACT

A number of related combinatorial properties of a cardinal κ contradicting AC are examined. Chief results include: (1) For many ordinals γ, $\kappa \to (\kappa)^{\gamma}$ implies $\kappa \to (\kappa)^{<\gamma}$. (2) For many ordinals γ, if $\kappa \to (\kappa)^{\gamma}_{\alpha}$ for all $\alpha < \kappa$, then κ is γ-weakly ineffable. (3) For all infinite cardinals γ, $\kappa \to (\kappa)^{<\gamma}$ implies κ is $<\gamma$-weakly ineffable.

In recent years there has been a growing interest in cardinals satisfying infinite-exponent partition relations. Despite their incompatibility with the full Axiom of Choice [9], the existence of such cardinals leads to many interesting results [3], [4], [7], [8]. The proofs of these are often distinctive and appear to have a flavor of their own. Drawing on few outside theorems and using methods constrained by lack of Choice, they build on each other and create a delicate and peculiar universe. At present these cardinals are chiefly obtainable through the Axiom of Determinateness (AD) [10], [11]. AD in fact implies $\aleph_1 \to (\aleph_1)^{\aleph_1}$, a property which, as we shall see, implies all those considered in this paper. Apart from this, the consistency relative to other axioms of the existence of these cardinals is unknown. As a measure of the difficulty Kleinberg has shown that $\kappa \to (\kappa)^{\omega+\omega}$ implies that the ω-closed, unbounded filter is a measure on κ, while Martin has shown that a consequence of this is the existence of models of ZF with very many measurables.

The most natural way to view cardinals satisfying infinite-exponent partition relations is as infinite generalizations of weakly compact cardinals. Consideration of this suggests infinite generalizations of similar but stronger cardinals: Ramsey and ineffable cardinals. The purpose of this paper is to make the necessary definitions and prove some general and specific results concerning such cardinals.

[†] Many of the results in this paper appeared originally in the author's doctoral dissertation.
Received March 13, 1977 and in revised form June 3, 1977

Section 1 is concerned with γ-Ramsey cardinals. The first and only previous result on these is due to E. M. Kleinberg [7] who proved:

THEOREM. *If* κ, γ *are any infinite ordinals, then*
(a) $\kappa \to (\kappa)^{\gamma+\gamma}$ *implies* $\kappa \to (\kappa)^{\gamma}_{2^\alpha}$ *for all* $\alpha < \kappa$, *and*
(b) $\kappa \to (\kappa)^{\gamma}_{2^\alpha}$ *implies* $\kappa \to (\kappa)^{<\gamma}$.

At first glance it seems apparent that the property of being γ-Ramsey ($\kappa \to (\kappa)^{<\gamma}$) is strictly weaker than satisfying $\kappa \to (\kappa)^{\gamma}$. The problem, however, is that with γ-Ramsey cardinals, we are forced to deal with γ-many partitions. Nevertheless, that general principle is almost always true, though the question is not yet closed. In this section we will prove for considerably many γ that $\kappa \to (\kappa)^{\gamma}$ does imply $\kappa \to (\kappa)^{<\gamma}$. The least γ which will fail to be covered by the theorems is the otherwise undistinguished ordinal, $\omega \cdot \omega + \omega$.

Section 2 is concerned with γ- and $< \gamma$-ineffability and weak ineffability. It will be easily seen from the definition that if κ is γ-weakly ineffable, then $\kappa \to (\kappa)^{\gamma}_{2^\alpha}$ for all $\alpha < \kappa$. Motivated, as in the previous section, by a desire to equate new cardinals with old, but using entirely different techniques, we will show that for many ordinals γ, $\kappa \to (\kappa)^{\gamma}_{2^\alpha}$ for all $\alpha < \kappa$ implies that κ is γ-weakly ineffable. We will also prove that for all cardinals κ, if κ is γ-Ramsey, then κ is $< \gamma$-weakly ineffable. We will show further that it is not often possible for κ to be γ- or $< \gamma$-ineffable.

§0. Definitions

In this paper, κ will always denote an uncountable cardinal. All other Greek letters may represent arbitrary ordinals. For any κ, γ, the set $[\kappa]^{\lambda}$ is the set of all subsets of κ of order-type γ. We will sometimes view a member of $[\kappa]^{\lambda}$ as a subset and sometimes as an increasing function from λ to κ. It will always be clear from the context, however, which meaning is intended.

An infinite ordinal γ with the property that $\alpha < \gamma$ implies $\alpha + \alpha < \gamma$ is said to be *indecomposable* (sometimes called a "power of ω").

Given any ordinal α and set A, a cardinal κ satisfies $\kappa \to (\kappa)^{\alpha}_{A}$ iff for all partitions F: $[\kappa]^{\alpha} \to A$, there is a set $X \subseteq \kappa$, $\bar{\bar{X}} = \kappa$ such that $\overline{F''[X]^{\alpha}} = 1$.

In this notation, $\kappa \to (\kappa)^{\alpha}_{2}$ may also be written $\kappa \to (\kappa)^{\alpha}$. If κ does not satisfy $\kappa \to (\kappa)^{\alpha}_{A}$, then we write $\kappa \not\to (\kappa)^{\alpha}_{A}$, and if F if a partition which fails to have a homogeneous set, we say it is a *bad partition.*

Certain facts are immediate from the definitions:

FACT 1. If κ is a cardinal satisfying $\kappa \to (\kappa)^{\alpha}$, and $\beta < \alpha$, then κ satisfies $\kappa \to (\kappa)^{\beta}$.

FACT 2. If a cardinal κ satisfies $\kappa \to (\kappa)^{\alpha}$, then κ satisfies $\kappa \to (\kappa)^{\alpha}_{n}$, for all n.

The proofs of these facts are elementary. For the first, note that $\alpha \geqq 2$ implies κ is regular by standard methods. Induction on n suffices for the second. Fact 1 implies among other things that κ is weakly compact, and hence regular.

We say that κ is γ-*Ramsey*, or that κ satisfies the relation $\kappa \to (\kappa)^{<\gamma}$ iff for all partitions $F\colon [\kappa]^{<\gamma} \to 2$, there is a set $X \subseteq \kappa$, $\bar{\bar{X}} = \kappa$ such that for all $\alpha < \gamma$, $\overline{F''[X]^{\alpha}} = 1$. X is said to be *homogeneous* for F. Note that by this definition a Ramsey cardinal is an ω-Ramsey cardinal.[†]

Given κ, γ, κ is said to be γ-*weakly ineffable* (resp. γ-*ineffable*) if given any collection $\{A_p\}_{p \in [\kappa]^{\gamma}}$ of subsets of κ such that for all $p \in [\kappa]^{\gamma}$, $A_p \subseteq p(0)$, then there exists a set $X \subseteq \kappa$, $\bar{\bar{X}} = \kappa$ (resp. X stationary) and set $A \subseteq \kappa$ such that for all $p \in [X]^{\gamma}$, $A_p = A \cap p(0)$. We call X the *cohering set* for $\{A_p\}_{p \in [\kappa]^{\gamma}}$. Two sets A_p and A_q such that $p(0) \leqq q(0)$ are said to *cohere* if $A_p = A_q \cap p(0)$. Note: X is a stationary subset of κ if X intersects all closed, unbounded subsets of κ.

For γ finite, γ-weak ineffability is a strictly stronger property than $\kappa \to (\kappa)^{\gamma}_{2^{\lambda}}$ for all $\lambda < \kappa$. With AC, it is shown that the least κ which is 1-ineffable, is greater than cardinals satisfying $\kappa \to (\kappa)^{n}_{2^{\lambda}}$ for all $n < \kappa$, $\lambda < \kappa$. Among the more popular large cardinals,[††] only the Ramsey cardinal is strong enough to be 1-weakly ineffable. Ineffable, or 1-ineffable cardinals were invented by Jensen and Kunen [6]. The hierarchy of n-ineffable cardinals was originally defined by Baumgartner. The relationship between these and other cardinals is detailed in his papers [1], [2].

Given κ, γ, κ is said to be $<\gamma$-weakly ineffable (resp. $<\gamma$-ineffable) if given any collection $\{A_p\}_{p \in [\kappa]^{<\gamma}}$ of subsets of κ such that for all $p \in [\kappa]^{<\gamma}$, $A_p \subseteq p(0)$, then there exists a set $X \subseteq \kappa$, $\bar{X} = \kappa$ (resp. X stationary) such that for all $\alpha < \gamma$, $p, q \in [X]^{\alpha}$, A_p and A_q cohere.

§1. γ-Ramsey cardinals and obliging ordinals

An ordinal γ is called *obliging* if it can be proved that $\kappa \to (\kappa)^{\gamma}$ implies $\kappa \to (\kappa)^{<\gamma}$, for all cardinals $\kappa > \gamma$.

[†] Note that for all κ, $\kappa \not\to (\kappa)^{<\kappa}$. The partition $F\colon [\kappa]^{<\kappa} \to 2$ defined by: if $p \in [\kappa]^{\alpha}$, $F(p) = 0$ iff $p(0) = \alpha$ provides a counterexample.

[††] For example, none of the following cardinal properties guarantee weak ineffability: strong inaccessibility, weak compactness, Mahlo, Jonsson, Rowbottom, α-Erdös (the least γ such that $\gamma \to (\alpha)^{<\omega}$).

Our first goal will be to show that all indecomposable ordinals are obliging. To this end, we require the following lemma:

LEMMA 1. *Given that an indecomposable ordinal γ is expressible as $\alpha \cdot \beta$, where α and β are limit ordinals less than γ, then γ is obliging.*

PROOF. Suppose we are given a partition $F: [\kappa]^{<\gamma} \to 2$. We make the following definitions: we shall say two sequences $p, q \in [\kappa]^\gamma$ are *similar* whenever $F(p \restriction \lambda) = F(q \restriction \lambda)$ for all $\lambda < \gamma$.

The set $\{\lambda < \gamma \mid F(p \restriction \lambda) = 0\}$ is called the *similarity type* of p.

If $p \in [\kappa]^\gamma$, let $p = \bigcup_{\delta<\beta} p_\delta$, where the $\{p_\delta\}_{\delta<\beta}$ are the successive blocks of p of length α, i.e., for all $\delta < \beta$, $p_\delta \in [p]^\alpha$ and if $\delta_1 < \delta_2$, then $\bigcup p_{\delta_1} \leqq \bigcap p_{\delta_2}$. If $q \in [\beta]^\beta$, let p_q be the element of $[p]^\gamma$ consisting of $\bigcup_{\delta \in q} p_\delta$.

We now define $G: [\kappa]^\gamma \to 2$ by: $G(p) = 0$ iff for all $q \in [\beta]^\beta$, p is similar to p_q. Since $\kappa \to (\kappa)^\gamma$, let $X \subseteq \kappa$, $\bar{X} = \kappa$ be homogeneous for G.

CLAIM. $G''[X]^\gamma = \{0\}$.

PROOF OF CLAIM. Let $H: [\kappa]^\beta \to 2^\gamma$ be defined by $H(q) =$ the similarity type of $\chi_q = \bigcup_{\delta \in q} \chi_\delta$, where $X = \bigcup_{\delta<\kappa} \chi_\delta$, $\chi_\delta \in [X]^\alpha$ and $\delta_1 < \delta_2 \to \bigcup \chi_{\delta_1} \leqq \bigcap \chi_{\delta_2}$. $p_q = \bigcup_{\delta \in q} p_\delta$.

Since $\beta + \beta < \gamma$, Fact 1 and Kleinberg's theorem imply $\kappa \to (\kappa)^\beta_{2^\gamma}$, hence there is a set $Y \subseteq \kappa$, $\bar{Y} = \kappa$, homogeneous for H. Let $q \in [Y]^\beta$. It follows that $\chi_q \in [X]^\gamma$ and that $G(\chi_q) = 0$, and hence $G''[X]^\gamma = \{0\}$.

We now claim that X is homogeneous for F: Suppose that $\lambda < \gamma$ and $p, q \in [X]^\lambda$. Let p' and q' extend p and q respectively such that $p', q' \in [X]^{\alpha \cdot \eta}$ for some $\eta < \beta$ (since β is a limit ordinal), and let $r \in [X]^\gamma$ be such that $\bigcup p', \bigcup q' \leqq \bigcap r$, and so $p'' = p' \cup r$ and $q'' = q' \cup r$ are in $[X]^\gamma$ by the indecomposability of γ. Then since $G''[X]^\gamma = \{0\}$ the similarity types of p'', q'' and r are the same, so

$$F(p) = F(p'' \restriction \lambda) = F(q'' \restriction \lambda) = F(q).$$

This completes the proof of Lemma 1. □

THEOREM 1. *For any ordinal γ, if γ is indecomposable, then γ is obliging.*

PROOF. Let $F: [\kappa]^{<\gamma} \to 2$ be any partition. For any $p \in [\kappa]^\gamma$ consider the following list of equations:

$$F(p(0)) = F(p(1)),$$

$$F(p(2), p(3)) = F(p(4), p(5)),$$

where in general, the βth equation for $\beta < \gamma$ is

$$F(p(\alpha_\beta), p(\alpha_\beta + 1), \cdots, p(\alpha_\beta + \delta), \cdots)_{\delta<\beta}$$
$$= F(p(\alpha_\beta + \beta), \cdots, p(\alpha_\beta + \beta + \delta), \cdots)_{\delta<\beta}$$

and where $\alpha_0 = 0$, $\alpha_1 = 2$, and

$$\alpha_{\beta+1} = \alpha_\beta + \beta \cdot 2,$$

$$\alpha_\lambda = \bigcup_{\beta<\lambda} \alpha_\beta \text{ for } \lambda \text{ a limit ordinal.}$$

We have two cases:

Case 1. We are unable to complete the list because for some $\beta < \delta$, $\alpha_\beta \geqq \gamma$.

Case 2. The length of p, γ, is large enough to complete this list.

First, case 1. Suppose we have used up all of p before the βth equation. Then $(\beta + \beta) \cdot \beta \geqq \gamma$. Let α be the least ordinal such that for some $\delta < \gamma$, $\alpha \cdot \delta \geqq \gamma$, and let δ be the least ordinal such that $\alpha \cdot \delta \geqq \gamma$. By the indecomposability of γ, δ is a limit ordinal, and hence so is α. Since $\alpha \cdot \lambda < \gamma$ for all $\lambda < \delta$, it then follows that $\alpha \cdot \delta = \gamma$. Lemma 1, then, shows that γ is obliging.

Now, case 2. We define a partition $G: [\kappa]^\gamma \to 2$ by:

for any $p \in [\kappa]^\gamma$, $G(p) = 0$ iff all the equations in the list are true. Let $X \in [\kappa]^\kappa$, $\bar{X} = \kappa$ be homogeneous for G.

Claim. $G''[X]^\gamma = \{0\}$.

Proof of Claim. Simply define a sequence $p \in [X]^\gamma$ such that $G(p) = 0$ as follows:

Let $p(0)$, $p(1)$ be the least elements of X such that $F(p(0)) = F(p(1))$ and continue in this way. To avoid any use of the Axiom of Choice, choose the members of p consecutively: suppose we have chosen enough elements of p to satisfy the first β equations for all $\beta < \alpha$. To satisfy the αth equation, take the first $\alpha \cdot 3$ members of X greater than all members of p chosen so far: $\delta_0, \delta_1, \cdots, \delta_\eta, \cdots_{\eta<\alpha\cdot 3}$ and consider $F(\delta_0, \cdots, \delta_\eta, \cdots)_{\eta<\alpha}$, $F(\delta_\alpha, \cdots, \delta_{\alpha+\eta}, \cdots)_{\eta<\alpha}$ and $F(\delta_{\alpha+\alpha}, \cdots, \delta_{\alpha+\alpha+\eta}, \cdots)_{\eta<\alpha}$. Two of these must be equal, and whichever they are, add the appropriate elements to p. When p is complete, all the equations are true, and $G(p) = 0$, proving the claim.

To complete the proof of the theorem, we observe that X, minus its first γ members constitutes a homogeneous set for F, for suppose for some $\alpha < \gamma$, $p, q \in [X - X(\gamma)]^\alpha$. Let $r \in [X]^\alpha$ be such that $\bigcup p, \bigcup q < r(0)$, and then form $s_1, s_2 \in [X]^\gamma$ such that for s_1, the αth equation in the list reads "$F(p) = F(r)$", and such that for s_2, the αth equation in the list reads "$F(q)=F(r)$". Since $G(s_1)=G(s_2)=0$, these equations are both true, and hence $F(p)=F(q)$. □

The situation where $\gamma=\alpha+\beta$, α, $\beta<\gamma$ is more difficult. The following results cover a number of cases, but by no means all.

THEOREM 2. *If γ is obliging, then $\gamma+n$ is obliging for all $n<\omega$.*

PROOF. By induction on n. If $\gamma+n-1$ is obliging, any partition F: $[\kappa]^{<\gamma+n}\to 2$ can be handled in two steps, first find $X\subseteq\kappa$, $\bar{X}=\kappa$ homogeneous for $F\restriction[\kappa]^{<\gamma+n-1}$, then find $Y\subseteq X$, $\bar{Y}=\kappa$ homogeneous for $F\restriction[\kappa]^{\gamma+n-1}$.

The next lemma and theorem can be viewed as an extension of Kleinberg's theorem.

LEMMA 2. *Suppose κ satisfies $\kappa\to(\kappa)^{\alpha+\beta}$, where α is obliging and $\beta\leqq\alpha$ is indecomposable. Then: $\kappa\to(\kappa)^{<\alpha+\beta}$ iff $\kappa\to(\kappa)^{\alpha}_{2^\beta}$.*

PROOF. Suppose κ satisfies $\kappa\to(\kappa)^{<\alpha+\beta}$. Then if we are given any partition F: $[\kappa]^{\alpha}\to 2^\beta$ we can define a partition G: $[\kappa]^{<\alpha+\beta}\to 2$ as follows: if $p\in[\kappa]^\lambda$, $\lambda<\alpha+\beta$, then: if $\lambda<\alpha$, $G(p)=0$, otherwise; if $\lambda=\alpha+\delta$ for some $\delta<\beta$, then $G(p)=1$ iff $\delta\in F(p\restriction\alpha)$.

If X is any set homogeneous for G, it must also be homogeneous for F. If $p,q\in[X]^\alpha$ and $\delta<\beta$, simply let $r\in[X]^\delta$ be such that $\bigcup p,\bigcup q\leqq\bigcap r$. Then $\delta\in F(p)$ iff $G(p\cup r)=1$ iff $G(q\cup r)=1$ iff $\delta\in F(q)$.

Going the other way, suppose F: $[\kappa]^{<\alpha+\beta}\to 2$ is any partition. Since α is obliging, let $X\subseteq\kappa$, $\bar{X}=\kappa$ be homogeneous for $F\restriction[\kappa]^{<\alpha}$. Next, define G: $[X]^{\alpha+\beta}\to 2$ by: if $p\cup q\in[X]^{\alpha+\beta}$ ($p\in[X]^\alpha$, $q\in[X]^\beta$, $\bigcup p\leqq\bigcap q$), then $G(p\cup q)=0$ iff q is homogeneous for p, that is, $G(p\cup q)=0$ iff for all $\lambda<\beta$ and for all $q',q''\in[q]^\lambda$, $F(p\cup q')=F(p\cup q'')$. Let $Y\subseteq X$ be homogeneous for F, $\bar{Y}=\kappa$.

CLAIM. $G''[Y]^{\alpha+\beta}=\{0\}$.

PROOF OF CLAIM. Choose any $p\in[X]^\alpha$ and define H: $[Y-\bigcup p]^{<\beta}\to 2$ by $H(q)=F(p\cup q)$ for all $q\in[Y-\bigcup p]^{<\beta}$. Since $\beta\leqq\alpha$, we have $\kappa\to(\kappa)^{\beta+\beta}$, and so by Kleinberg's theorem, there is a set $Z\subseteq Y-\bigcup p$ homogeneous for H. Then, if $q\in[Z]^\beta$, then $G(p\cup q)=0$, hence $G''[Y]^{\alpha+\beta}=\{0\}$.

CLAIM. For any $p\in[Y]^\alpha$, $Y-\bigcup p$ is homogeneous for F with p, that is, if $\delta<\beta$ and $s_1,s_2\in[Y-\bigcup p]^\delta$, then $F(p\cup s_1)=F(p\cup s_2)$.

PROOF OF CLAIM. Given any such p,s_1,s_2 let $s_3\in[Y-\bigcup p]^\delta$ be such that $\bigcup s_1,\bigcup s_2<s_3(0)$. Since β is indecomposable, let $q_1,q_2\in[Y-\bigcup p]^\beta$ be such

that $s_1 \cup s_3 \subseteq q_1$, $s_2 \cup s_3 \subseteq q_2$. Then since $G(p \cup q_1) = G(p \cup q_2) = 0$, we have $F(p \cup s_1) = F(p \cup s_3) = F(p \cup s_2)$, proving the claim.

As a consequence of this, for any $p \in [Y]^\alpha$, there is associated a subset $I(p)$ of β defined by:

$$\delta \in I(p) \text{ iff for all } q \in [Y - \bigcup p]^\delta,\ F(p \cup q) = 1.$$

This defines a partition $I: [Y]^\alpha \to 2^\beta$. Let $Z \subseteq Y$, $\bar{Z} = \kappa$, be homogeneous for I.

Claim. Z is homogeneous for F.

Proof of Claim. Suppose $s_1, s_2 \in [Z]^{<\alpha+\beta}$. If $\bar{s}_1 = \bar{s}_2 < \alpha$, then $F(s_1) = F(s_2)$ by the homogeneity of X. If $\bar{s}_1 = \bar{s}_2 \geqq \alpha$, and $p_1 = s_1 \restriction \alpha$, $p_2 = s_2 \restriction \alpha$, let q_1, q_2 be such that $p_1 \cup q_1 = s_1$ and $p_2 \cup q_2 = s_2$, $\bigcup p_1 \leqq \bigcap q_1$, $\bigcup p_2 \leqq \bigcap q_2$ and let $\delta = \bar{q}_1 = \bar{q}_2$. Then,

$$F(s_1) = 1 \text{ iff } \delta \in I(p_1) \text{ iff } \delta \in I(p_2) \text{ iff } F(s_2) = 1.$$

This completes the proof of the lemma. □

Theorem 3. *If γ is indecomposable, then for all $n \geqq 1$, $\gamma \cdot n$ is obliging.*

Proof. The proof is by induction on n. The case for $n = 1$ is covered by Theorem 1.

Suppose $\gamma \cdot n$ is obliging, $n < \omega$. By the previous lemma, to prove $\gamma \cdot (n+1)$ obliging, it suffices to show that $\kappa \to (\kappa)^{\gamma \cdot n}_{2^\gamma}$. In the interests of generality, we will show $\kappa \to (\kappa)^{\gamma \cdot n}_{2^\alpha}$ for all $\alpha < \kappa$ in two stages. Our first step will be to show that $\kappa \to (\kappa)^{\gamma \cdot n}_{\alpha}$ for all $\alpha < \kappa$ is a consequence of $\kappa \to (\kappa)^{\gamma \cdot (n+1)}$.

Suppose $F: [\kappa]^{\gamma \cdot n} \to \alpha$ is any partition. Let $G: [\kappa]^{\gamma \cdot n + \gamma} \to 3$ be the following partition: if $p_0, p_1, \cdots, p_n \in [\kappa]^\gamma$, and $\bigcup p_i \leqq \bigcap p_{i+1}$ for all $i < n$, then:

$$G(p_0 \cup \cdots \cup p_n) = \begin{cases} 0 & \text{iff } F(p_0 \cup \cdots \cup p_{n-1}) = F(p_1 \cup \cdots \cup p_n) \\ 1 & \text{iff } F(p_0 \cup \cdots \cup p_{n-1}) > F(p_1 \cup \cdots \cup p_n) \\ 2 & \text{iff } F(p_0 \cup \cdots \cup p_{n-1}) < F(p_1 \cup \cdots \cup p_n). \end{cases}$$

By Fact 2, $\kappa \to (\kappa)^{\gamma \cdot n + \gamma}_3$. Let $X \subseteq \kappa$, $\bar{X} = \kappa$ be homogeneous for G.

Claim. $G''[X]^{\gamma \cdot n + \gamma} = \{0\}$.

Proof of Claim. Clearly $G''[X]^{\gamma \cdot n + \gamma} \neq \{1\}$ for we could then choose for all $n < \omega$, $p_n \in [X]^\gamma$ such that $\bigcup p_n < \bigcap p_{n+1}$ and then

$$F(p_0 \cup \cdots \cup p_{n-1}) > F(p_1 \cup \cdots \cup p_n) > \cdots > F(p_q \cup \cdots \cup p_{n+q-1}) > \cdots$$

an impossibility.

Suppose $G''[X]^{\gamma\cdot n+\gamma}=\{2\}$. Let $\{q_\delta\}_{\delta<\alpha+1}$ be successive sequences from $[X]$ of length $\gamma\cdot n$, i.e., for all $\delta<\alpha+1$, $q_\delta\in[X]^{\gamma\cdot n}$ and $\delta<\delta'<\alpha+1$ implies $\bigcup q_\delta<\bigcap q_{\delta'}$. For each $\delta<\alpha+1$, let $p_{\delta,0},p_{\delta,1},\cdots,p_{\delta,n-1}$, be the successive γ sequences composing q_δ. Then for all $\delta<\delta'<\alpha+1$,

$$\begin{aligned}F(q_\delta)&=F(p_{\delta,0}\cup p_{\delta,1}\cup\cdots\cup p_{\delta,n-1})\\&<F(p_{\delta,1}\cup\cdots\cup p_{\delta,n-1}\cup p_{\delta',0})\\&\cdots\\&<F(p_{\delta',0}\cup\cdots\cup p_{\delta',n-1})\\&=F(q_{\delta'}).\end{aligned}$$

Thus, $\{F(q_\delta)\}_{\delta<\alpha}$ forms an increasing α-sequence, and $F(q_\alpha)\geqq\alpha$, an impossibility, proving the claim.

We now claim that X is homogeneous for F, for if $p,q\in[X]^{\gamma\cdot n}$, let $r\in[X]^{\gamma\cdot n}$ be such that $\bigcup p,\bigcup q<\bigcap r$. Once again, let $p_0,p_1,\cdots,p_{n-1}$ and $r_0,r_1,\cdots,r_{n-1}$ be the successive γ sequences of p and r respectively. Since $G''[X]^{\gamma\cdot n+\gamma}=\{0\}$,

$$\begin{aligned}F(p)&=F(p_0\cup\cdots\cup p_{n-1}\cup r_0)\\&=F(p_1\cup\cdots\cup p_{n-1}\cup r_0\cup r_1)\\&=\cdots\\&=F(r_0\cup\cdots\cup r_{n-1})\\&=F(r).\end{aligned}$$

Similarly, $F(q)=F(r)$ and X is homogeneous for F.

Using the relation, $\kappa\to(\kappa)^{\gamma\cdot n}_\alpha$ for all $\alpha<\kappa$ we will now show $\kappa\to(\kappa)^{\gamma\cdot n}_{2^\alpha}$.

Suppose $F:[\kappa]^{\gamma\cdot n}\to 2^\alpha$ to be any partition, $\alpha<\kappa$. Let $G:[\kappa]^{\gamma\cdot n}\to\alpha$ be defined as follows: if $p\in[\kappa]^{\gamma\cdot n}$, then

$$G(p)=\begin{cases}0 & \text{if } F(p)=F(q),\text{ for all } q\in[p]^{\gamma\cdot n},\\ \beta+1 & \text{if } \beta \text{ is the least ordinal such that for some } q\in[p]^{\gamma\cdot n},\ \beta \text{ is in one, but not both of the sets } F(p),\ F(q).\end{cases}$$

Let $X\subseteq\kappa$, $\bar{X}=\kappa$, be homogeneous for G.

CLAIM. $G''[X]^{\gamma\cdot n}=\{0\}$.

PROOF OF CLAIM. Suppose $G''[X]^{\gamma\cdot n}=\{\beta+1\}$. Define $H:[X]^{\gamma\cdot n}\to 2$ by

$H(p) = 1$ iff $\beta \in F(p)$, for all $p \in [X]^{\gamma \cdot n}$. Let $Y \subseteq X$, $\bar{Y} = \kappa$, be homogeneous for H. It is clear that for any $p \in [Y]^{\gamma \cdot n}$ and $q \in [p]^{\gamma \cdot n}$,

$$F(p) \cap \beta = F(q) \cap \beta \qquad \text{by the homogeneity of } X,$$
$$F(p) \cap (\beta + 1) = F(q) \cap (\beta + 1) \quad \text{by the homogeneity of } Y,$$

hence $G(p) > \beta + 1$, a contradiction, and the claim is proved.

Suppose now that $p, q \in [X]^{\gamma \cdot n}$ are such that the consecutive supremums of their consecutive γ-sequences are the same, i.e., that $p = p_0 \cup \cdots \cup p_{n-1}$ and $q = q_0 \cup \cdots \cup q_{n-1}$, $p_i, q_i \in [X]^{\gamma}$, and that $\bigcup p_i = \bigcup q_i$, for all i. It then follows that $F(p) = F(q)$, because each $p_i \cup q_i$ is a member of $[X]^{\gamma}$ (a consequence of γ's indecomposability) and so $p \cup q \in [X]^{\gamma \cdot n}$. Thus $G(p \cup q) = 0$ implying that $F(p) = F(p \cup q) = F(q)$. Hence, we may make the following unambiguous definition: $H\colon [\kappa]^n \to 2^{\alpha}$ is defined by: for all $\langle \beta_0, \beta_1, \cdots, \beta_{n-1} \rangle \in [\kappa]^n$,

$$H(\langle \beta_0, \beta_2, \cdots, \beta_{n-1} \rangle) = \begin{cases} A & \text{if for all } i < n \text{ there is a } p_i \in [X]^{\gamma}, \\ & \text{with } \bigcup p_i = \beta_i < \bigcap p_{i+1} \text{ and} \\ & F(p_0 \cup \cdots \cup p_{n-1}) = A \\ \varnothing & \text{otherwise.} \end{cases}$$

Since $[\kappa]^n$ is well-orderable, $H''[\kappa]^n$ is well-ordered. By the above arguments, $H''[\kappa]^n \supseteq F''[X]^{\gamma \cdot n}$. By a well-known theorem, there can be no κ sequence of distinct elements of 2^{α} (the proof requires no more than $\kappa \to (\kappa)^4$, see [7]), thus the cardinality of $F''[X]^{\gamma \cdot n}$ is $\beta < \kappa$. Let $f\colon F''[X]^{\gamma \cdot n} \to \beta$ be a one-one map. Since $\kappa \to (\kappa)_{\beta}^{\gamma \cdot n}$, there is a homogeneous set $Y \subseteq X$, $\bar{Y} = \kappa$ such that Y is homogeneous for $f \circ F$. Y is then homogeneous for F. This completes the proof of Theorem 3. □

As a corollary, we have:

COROLLARY.[†] *For all indecomposable* γ,
1) $\kappa \to (\kappa)^{\gamma \cdot (n+1)}$ *implies* $\kappa \to (\kappa)_{\beta}^{\gamma \cdot n}$ *for all* $\beta < \kappa$, *and*
2) $\kappa \to (\kappa)_{\beta}^{\gamma \cdot n}$ *for all* $\beta < \kappa$ *implies* $\kappa \to (\kappa)_{2^{\beta}}^{\gamma \cdot n}$ *for all* $\beta < \kappa$.

The hypothesis that $\alpha < \gamma$ implies $\alpha + \alpha < \gamma$ was used only once, and it seems avoidable, though not without difficulty or new techniques.

The least infinite ordinal not covered by Theorems 1, 2, and 3 is $\omega \cdot \omega + \omega$.

[†] Baumgartner has pointed out to me that the techniques of the previous proof can be cleverly expanded to yield: "if λ is a limit ordinal, then $\kappa \to (\kappa)_{\alpha}^{\lambda}$ implies $\kappa \to (\kappa)_{2^{\alpha}}^{\lambda}$". The restriction that λ is a limit can also be eliminated.

Since proving this obliging will require proving that $\kappa \to (\kappa)^{\omega\cdot\omega+\omega}$ implies $\kappa \to (\kappa)^{\omega\cdot\omega}_{\omega}$, it does not look easy.

In closing this section, it should be noted that a simple application of the techniques used above will produce results of the following sort: If γ is sufficiently large (a cardinal, for example, or a Δ_0-admissable, etc.) then $\kappa \to (\kappa)^{<\gamma}$ implies $\kappa \to (\kappa)^{<\gamma}_{\alpha}$ for all $\alpha < \gamma$, where this latter property is defined in the obvious way.

§2. γ- and $<\gamma$-ineffable cardinals

The property of being γ-weakly ineffable implies $\kappa \to (\kappa)^{\gamma}_{\alpha}$ for all $\alpha < \kappa$, and in fact implies $\kappa \to (\kappa)^{\gamma}_{2^{\alpha}}$ for all $\alpha < \kappa$, for if we are given a partition $F\colon [\kappa]^{\gamma} \to 2^{\alpha}$, we can define for all $p \in [\kappa - \alpha]^{\gamma}$, $A_p = F(p)$, and it follows that any cohering set for the A_p must be homogeneous for F. Thus, if we are to prove a cardinal κ to be γ-weakly ineffable, we must start with at least $\kappa \to (\kappa)^{\gamma}_{2^{\alpha}}$ for all $\alpha < \kappa$.

THEOREM 4. *If $\kappa \to (\kappa)^{\gamma}_{2^{\alpha}}$ for all $\alpha < \kappa$ and $\omega\cdot\gamma = \gamma$, then κ is γ-weakly ineffable.*

PROOF. Some necessary notation first. For any sequence $p \in [\kappa]^{\gamma}$, let ${}_{\omega}p$ be the sequence consisting of the consecutive ω-sups of p, i.e., since $\omega\cdot\gamma = \gamma$, let $p = p_0 \cup p_1 \cup \cdots \cup p_{\delta} \cup \cdots_{\delta<\gamma}$ where $p_{\delta} \in [p]^{\omega}$ for each $\delta < \gamma$, and where $\delta < \delta' < \gamma$ implies $\bigcup p_{\delta} \leqq \bigcap p_{\delta'}$.

Then ${}_{\omega}p$ is the sequence, $\bigcup p_0, \bigcup p_1, \bigcup p_2, \cdots$ or, ${}_{\omega}p(\delta) = \bigcup p_{\delta}$ for all $\delta < \gamma$.

Now, suppose we are given a collection $\{A_p\}_{p\in[\kappa]^{\gamma}}$ of subsets of κ such that for all $p \in [\kappa]^{\gamma}$, $A_p \subseteq p(0)$. For any $\alpha < \kappa$, let $F_{\alpha}\colon [\kappa]^{\gamma} \to 2^{\alpha}$ be the partition:

$$F_{\alpha}(p) = A_{{}_{\omega}p} \cap \alpha, \quad \text{for all } p \in [\kappa]^{\gamma}.$$

By $\kappa \to (\kappa)^{\gamma}_{2^{\alpha}}$, there is a homogeneous set X for F. Let $A_{\alpha} = F''_{\alpha}[X]^{\gamma}$. A_{α} does not depend on the choice of X, for if X_1 and X_2 are both homogeneous, we could choose $p_1 \in [X_1]^{\gamma}$, $p_2 \in [X_2]^{\gamma}$ such that for all $\delta, p_1(\delta) < p_2(\delta) < p_1(\delta+1)$. In this way, ${}_{\omega}p_1 = {}_{\omega}p_2$, and hence $F''_{\alpha}[X_1]^{\gamma} = F''_{\alpha}[X_2]^{\gamma}$.

Now we define $F\colon [\kappa]^{\gamma} \to 2$ by

$$F(p) = 0 \text{ iff } A_{{}_{\omega}p} \cap p(0) = A_{p(0)} \text{ for all } p \in [\kappa]^{\gamma}.$$

Let X be homogeneous for F.

CLAIM. $F''[X]^{\gamma} = \{0\}$.

Proof of Claim. Let α be the least element of X. Let X_α be homogeneous for F_α, and choose $p \in [X-\alpha]^\gamma$, $q \in [X_\alpha - \alpha]^\gamma$ such that ${}_\omega p = {}_\omega q$. Then we have: $F_\alpha(q) = A_\alpha$,

$$\text{so } A_{{}_\omega q} \cap \alpha = A_\alpha$$

$$\text{so } A_{{}_\omega p} \cap \alpha = A_\alpha$$

$$\text{so } F(\{\alpha\} \cup p) = 0,$$

and hence the claim is proved.

Next consider $\{A_\alpha \mid \alpha \in X\}$. We claim that these sets all cohere, that is, if $\alpha, \beta \in X$, $\alpha < \beta$, then $A_\alpha = A_\beta \cap \alpha$. To see this, simply let $p \in [X]^\gamma$ be such that $p(0) = \alpha$, $p(1) = \beta$ and let $q = p - p(0)$. Then

$$\begin{aligned} A_\beta \cap \alpha &= (A_{{}_\omega q} \cap q(0)) \cap \alpha \\ &= A_{{}_\omega q} \cap \alpha \\ &= A_{{}_\omega p} \cap p(0) \\ &= A_\alpha. \end{aligned}$$

We may thus define the set $A \subseteq \kappa$ by:

$$\alpha \in A \text{ iff } \alpha \in A_\beta \text{ for all } \beta > \alpha,\ \beta \in X.$$

Let $Y = {}_\omega X$.

Claim. Y and A satisfy the definition of γ-weak ineffability.

Proof of Claim. Suppose $p \in [Y]^\gamma$. Then there is a $q \in [X]^\gamma$ such that ${}_\omega q = p$. Then for any $\delta < p(0)$,

$$\begin{aligned} \delta \in A_p &\text{ iff } \delta \in A_p \cap q(n) \text{ for some } n \left(\text{since } \bigcup_{n<\omega} q(n) = p(0)\right) \\ &\text{ iff } \delta \in A_{{}_\omega q} \cap q(n) \text{ for some } n \\ &\text{ iff } \delta \in A_{q(n)} \text{ for some } n \text{ (applying the homogeneity} \\ &\qquad \text{of } X \text{ for } F \text{ to the sequence } q - q(n-1)) \\ &\text{ iff } \delta \in A \cap p(0). \end{aligned}$$

This completes the proof of the theorem. □

The situation with ineffability, as opposed to weak ineffability, is less encouraging:

THEOREM 5. *Given κ, γ ordinals, if γ is greater than all regular cardinals below κ, then κ is not γ-ineffable.*

PROOF. For each $p \in [\kappa]^\gamma$, let

$$A_p = \begin{cases} \{0\} \text{ if } p \text{ contains a limit point of itself,} \\ \{1\} \text{ otherwise.} \end{cases}$$

We will show that $\{A_p\}_{p \in [\kappa]^\gamma}$ cannot have a stationary cohering set, for if X is stationary, and A is such that for all $p \in [X]^\gamma$, $A_p = A \cap p(0)$, then A must be either $\{0\}$ or $\{1\}$.

Clearly, A cannot be $\{0\}$, since a $p \in [X]^\gamma$ can easily be found containing no limit points. On the other hand, consider (X) = the set of limit points of X. (X) is a closed, unbounded set, hence $X \cap (X) \neq \emptyset$. Let $\alpha \in X \cap (X)$. By hypothesis, there is a sequence of points in X of length less than γ with sup equal to α, thus there is a $p \in [X]^\gamma$ such that $A_p = \{0\}$. This proves the theorem. □

Since $< \gamma$ ineffability implies δ ineffability for all $\delta < \gamma$, a similar result holds for $< \gamma$ ineffability.

Our last theorem concerns $< \gamma$-weak ineffability:

THEOREM 6. *If $\gamma \geqq \omega$ is a cardinal, then $\kappa \to (\kappa)^{<\gamma}$ implies κ is $< \gamma$-weakly ineffable.*

PROOF. Suppose we are given sets $\{A_p\}_{p \in [\kappa]^{<\gamma}}$ such that for all $p \in [\kappa]^{<\gamma}$ $A_p \subseteq p(0)$. For every $\sigma \in [\gamma]^{<\omega}$, we will define a partition F_σ: $[\kappa]^\alpha \to 2$ for some $\alpha < \gamma$. Since γ is a cardinal, these partitions can all be coded up into one giant partition F: $[\kappa]^{<\gamma} \to 2$, such that any set X homogeneous for F will be homogeneous for all the $F_\sigma, \sigma \in [\gamma]^{<\omega}$. We will show how an appropriately chosen homogeneous set for F is a cohering set for the A_p.

We start by defining F_α: $[\kappa]^{\alpha \cdot 2} \to 2$, for $\alpha < \kappa$, by: if $p \in [\kappa]^\alpha$, $q \in [\kappa - \bigcup p]^\alpha$, $F_\alpha(p \cup q) = 0$ iff A_p and A_q cohere. Our method will be to pick a homogeneous set for the partitions with the least possible first element. By adding to this partition we will show that if the sets fail to cohere, there must be a homogeneous set with an even smaller first element. The new homogeneous set will be obtained with the functions f_α: $[\kappa]^{\alpha \cdot 2} \to \kappa$, $\alpha < \kappa$ defined by: if $p \in [\kappa]^\alpha$, $q \in [\kappa - \bigcup p]^\alpha$,

$$f_\alpha(p \cup q) = \begin{cases} p(0) \text{ if } A_p, A_q \text{ cohere;} \\ \text{the least } \delta < p(0) \text{ such that either} \\ \delta \in A_p - A_q \text{ or } \delta \in A_q - A_p, \text{ otherwise.} \end{cases}$$

Given any possible homogeneous set $X \in [\kappa]^\kappa$, $X = \{\beta_1, \beta_2, \cdots\}$, let $p = X \restriction \alpha$, $q = (X - p) \restriction \alpha$. If A_p and A_q do not cohere, then $f_\alpha(p \cup q) < \beta_1$. We would like to build a new homogeneous set with $f_\alpha(p \cup q)$ as the first element. We will carve this set out of $f''_\alpha[X]^{\alpha \cdot 2}$. To guarantee that it will also be homogeneous, we must expand our partition. For every $\sigma \in [\gamma]^{<\omega}$, $\sigma = \alpha_1, \alpha_2, \cdots \alpha_\kappa$ we define a partition $F_\sigma: [\kappa]^\alpha \to 2$, where $\alpha = \alpha_k \cdot 2 \cdot \alpha_{k-1} \cdot 2 \cdot \cdots \cdot \alpha_1 \cdot 2$ as follows: if $p \in [\kappa]^\alpha$ and $p_1, p_2, \cdots, p_\beta, \cdots_{\beta < \alpha'}$ are the consecutive components of p of length $\alpha_k \cdot 2$ where $\alpha' = \alpha_{k-1} \cdot 2 \cdot \ldots \cdot \alpha_1 \cdot 2$ (i.e., for each $\beta < \alpha'$, $p_\beta(0) \geqq \bigcup_{\delta < \beta} \cup p_\delta$, and $p = p_0 \cup p_1 \cup \cdots \cup p_\beta \cup \cdots_{\beta < \alpha'}$), then

$$F_\sigma(p) = F_{\sigma \restriction k-1}(f_{\alpha_k}(p_1), f_{\alpha_k}(p_2), \cdots, f_{\alpha_k}(p_\beta), \cdots)_{\beta < \alpha'}.$$

Now let $F: [\kappa]^{<\gamma} \to 2$ be a partition coding all the $\{F_\sigma\}_{\sigma \in [\gamma]^{<\omega}}$. Let β_1 be the least cardinal such that there exists a set X homogeneous for F, with $\beta_1 \in X$. Let $X \in [\kappa]^\kappa$ be such that $\beta_1 \in X$ and X is homogeneous for F.

Claim. $F''_\alpha[X]^{\alpha \cdot 2} = \{0\}$ for all $\alpha < \kappa$.

Proof of Claim. Suppose not. Then for some α, A_p and A_q don't cohere, where $p = X \restriction \alpha$ and $q = (X - \bigcup p) \restriction \alpha$, hence $f_\alpha(p \cup q) < \beta_1$. We will define a set $Y \subseteq f''_\alpha[X]^{\alpha \cdot 2}$, $Y = \{\alpha_1, \alpha_2, \cdots\}$ as follows: $\alpha_1 = f_\alpha(p \cup q)$. To define α_λ, for $\lambda < \kappa$, let $\beta \geqq \lambda$ be the sup of all ordinals used in defining α_δ for $\delta < \lambda$. Let $\rho = \bigcup_{\delta < \lambda} \alpha_\delta$, and let $p_1, p_2, \cdots, p_\eta, \cdots_{\eta < \kappa}$ be the successive α-sequences of $X - \beta$. Consider the κ-sequence of subsets of ρ:

$$\rho \cap A_{p_1}, \rho \cap A_{p_2}, \cdots, \rho \cap A_{p_\eta}, \cdots_{\eta < \kappa}.$$

As in the proof of Theorem 3, there can be no κ-sequence of distinct subsets of ρ, so at least two of these must be equal.

Since at least two of the sets are equal, choose the least such two, say $\rho \cap A_{p_{\delta_1}}$ and $\rho \cap A_{p_{\delta_2}}$ and let $\alpha_\lambda = f_\alpha(p_{\delta_1} \cup p_{\delta_2}) \geqq \rho$. By the construction of our partition, Y must also be homogeneous for F — note simply that for all $\sigma \in [\gamma]^{<\omega}$, $F''_\sigma[Y]^\beta \subseteq F''_{\sigma'}[X]^{\alpha \cdot 2 \cdot \beta}$, where σ' is the sequence σ with α added at the end. Finally, since the least element of Y is less than β_1, we have a contradiction, and the claim is proved.

Essentially we are now done. For every $\alpha < \gamma$, if $p, q \in [X]^\alpha$, $p(0) \leqq q(0)$, then there is an $r \in [X]^\alpha$ such that $\bigcup p, \bigcup q < r(0)$, and since $F_\alpha(p \cup r) = F_\alpha(q \cup r) = 0$, we have:

A_p and A_r cohere, or $A_p = A_r \cap p(0)$ and

A_q and A_r cohere, or $A_q = A_r \cap q(0)$

so $A_p = A_q \cap p(0)$ and A_p and A_q cohere.

This shows that X is a cohering set for the A_p, and completes the proof of the theorem. □

The hypothesis that γ is a cardinal may be weakened to: "there exists a one-one function $f: [\gamma]^{<\omega} \to \gamma$ such that for all $\alpha_1 < \alpha_2 < \cdots < \alpha_\kappa \in \gamma$, $f(\alpha_1, \alpha_2, \cdots, \alpha_k) \geqq \alpha_k \cdot 2 \cdot \alpha_{k-1} \cdot 2 \cdot \cdots \cdot \alpha_1 \cdot 2$", the sole purpose of such a map is to enable the partitions $\{F_\sigma\}_{\sigma \in [\gamma]^{<\omega}}$ to be coded by one partition F.

The case for $\gamma = \omega$ was first proved by Baumgartner [2]. This proof was based on that appearing in [5].

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